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Proper orientation of cacti ☆

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Abstract

An orientation of a graph G is *proper* if two adjacent vertices have different in-degrees. The *proper-orientation number* $\overrightarrow{\chi}(G)$ of a graph G is the minimum maximum in-degree of a proper orientation of G.

In [1], the authors ask whether the proper orientation number of a planar graph is bounded.

We prove that every cactus admits a proper orientation with maximum in-degree at most 7. We also prove that the bound 7 is tight by showing a cactus having no proper orientation with maximum in-degree less than 7. We also prove that any planar claw-free graph has a proper orientation with maximum in-degree at most 6 and that this bound can also be attained.

Keywords: proper orientation, graph coloring, cactus graph, claw-free graph, planar graph, block graph.

1. Introduction

For basic notions and notations on Graph Theory and Computational Complexity, the reader is referred to [2, 3]. All graphs in this work are considered to be simple.

An *orientation* D of a graph $G = (V, E)$ is a digraph obtained from G by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *in-degree* of v in D, denoted by $d^-_D(v)$, is the number of arcs with root v in D. We use the notation $d^-(v)$ when the orientation D is clear from the context. An orientation D of G is *proper* if $d^-(u) \neq d^-(v)$, for all $uv \in E(G)$. An orientation with maximum in-degree at most k is called a *k-orientation.* The *proper-orientation number* of a graph

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G, denoted by $\overrightarrow{\chi}(G)$, is the minimum integer k such that G admits a proper k-orientation.

This graph parameter was introduced by Ahadi and Dehghan [4]. They observed that this parameter is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation. Note that every proper orientation of a graph G induces a proper vertex coloring of G. Hence, we have the following sequence of inequalities:

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G).$$

These inequalities are best possible since, for a complete graph K_n:

$$\omega(K_n) - 1 = \chi(K_n) - 1 = \overrightarrow{\chi}(K_n) = \Delta(K_n) = n - 1.$$

In [4], the authors characterize the proper-orientation number of regular bipartite graphs, study other particular subclasses of regular graphs and prove the NP-hardness of the problem even when restricted to planar graphs.

Recently, it has been shown that the problem remains NP-hard for subclasses of planar graphs that are also bipartite and of bounded degree [1]. In the same paper, it is proved that the proper-orientation number of a tree is at most 4.

Theorem 1 ([1])**.** *Every tree has proper-orientation number at most* 4*.*

A natural question is to ask how this theorem can be generalized.

Problem 2. *Which graph classes containing the trees have bounded proper-orientation number ?*

In [1], several generalizations are suggested: on the one hand, the authors ask whether the proper-orientation number of planar graphs is bounded; on the other hand, they asked whether the proper-orientation number can be bounded by a function of the treewidth. We pose a similar, but simpler, question.

Problem 3. *Is there a constant c such that $\overrightarrow{\chi}(G) \leq c$, for every outerplanar graph G?*

Already this question seems highly non-trivial. One of the reasons is that, contrary to many other parameters like the chromatic number, the proper-orientation number is not monotonic. Recall that a graph parameter γ is *monotonic* if $\gamma(H) \leq \gamma(G)$ for every (induced) subgraph H of G. For example, the tree T^*, depicted in Figure 1, satisfies $\overrightarrow{\chi}(T^*) = 2$, while $\overrightarrow{\chi}(T^* \setminus \{x\}) = 3$ as $T^* \setminus \{x\}$ is exactly the tree T_3 mentioned in [1]. Its non-monotonicity makes it difficult to handle the proper-orientation number.

In this paper, we consider a standard graph class containing the trees, namely the cacti. A graph G is a *cactus* if every 2-connected component of G is either an edge or a cycle. Clearly, every cactus is an outerplanar graph. We prove that the proper orientation of such graphs is bounded by 7.

Theorem 4. *If G is a cactus, then $\overrightarrow{\chi}(G) \leq 7$.*

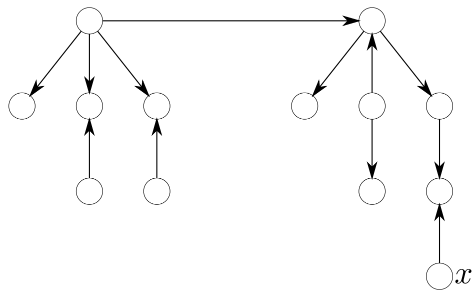


Figure 1: Tree T^* and a proper 2-orientation of it.

Furthermore, we show in Corollary 20 that this upper bound 7 is attained.

We conclude this section by introducing some definitions and previous results that we need in different sections of this work.

Let $S \subseteq V(G)$ be a subset of vertices of G and $F \subseteq E(G)$ be a subset of its edges. We denote by $G[S]$ the subgraph of G induced by S, by $G \backslash F$ the graph obtained from G by removing the edges in F from its edge set $E(G)$, and by $G - S$ the graph $G[V(G) \setminus S]$.

For any two adjacent vertices u and v of G, the edge (u, v) is denoted by uv. Given an orientation D of G, we denote the orientation of uv towards v by (u, v).

Let T be a tree. A *leaf* of T is a vertex with degree 1. A *twig* of T is a vertex which is not a leaf and whose neighbors are all leaves except possibly one. A *bough* of T is a vertex which is neither a leaf nor a twig and whose neighbors are all leaves or twigs except possibly one. A *branch* of T is a vertex which is neither a leaf nor a twig nor a bough and whose neighbors are all leaves or twigs or boughs except possibly one.

The definitions above are the same as the ones used in [1] and we borrow from them. Let G be a graph. The *block tree* associated to G is the tree $T(G)$ with vertex set the set of blocks of G such that two vertices are adjacent in $T(G)$ if and only if the blocks intersect. A block of order i is said to be an *i-block*. A *leaf block* (resp. *twig block*, *bough block*, *branch block*) is a block which is a leaf (resp. twig, bough, branch) in $T(G)$. By the definitions in the previous paragraph, observe that if B is of one of these types of blocks, then B may have a neighbor in $T(G)$ that is an exception in its neighborhood. If such a neighbor B' exists and $u \in B$ separates B from B', then we call u the *root* of B. Otherwise, we pick any vertex of B to be the root of B. If B is a twig block with root r, then the *twig subgraph of G with root r* is the union of B and all leaf blocks with root in $V(B) \setminus \{r\}$. If B is a bough block with root r, then the *bough subgraph of G with root r* is the union of B and all twig subgraphs with root in $V(B) \setminus \{r\}$. Observe that twig and bough subgraphs are connected.

Let B be a block in G. For any vertex $v \in B$ we denote by $G_B\langle v\rangle$ the connected component of $G \setminus E(B)$ containing v. If the block B is clear from the context, we often drop the subscript B.

2. Proper 7-orientation of cacti

In this section, we prove Theorem 4 by considering a minimum counter-example. Such a counter-example is a cactus G that admits no proper 7-orientation, and such that every cactus H with fewer vertices than G has a proper 7-orientation. Observe that such a counter-example G is clearly a connected graph.

The idea of the proof is to analyse the structure of the leaf, twig and bough subgraphs of G and observe that there is always one such subgraph in G with root r such that any proper 7-orientation of $G\langle r\rangle$ (which exists by the minimality of G) can be extended in a proper 7-orientation of G, which is a contradiction.

If B is a block of G with vertex set $\{v_1, \ldots, v_p\}$ appearing in this order on the cycle (or edge), then we write B as $\langle v_1, \ldots, v_p\rangle$.

Lemma 5. *Let $P = (v_1, \ldots, v_n)$ be a path on n vertices, $n \neq 2$. Then, there exists a proper 2-orientation of P such that v_1 and v_n have in-degree 0.*

Proof. If n is odd, it suffices to orient the arcs of P from vertices with odd indices towards vertices with even indices. This yields an alternating in-degree sequence of 0's and 2's that starts and ends with 0. If n is even, orient $(v_1, \ldots, v_{n-1})$ as above and $v_{n-1}v_n$ towards v_{n-1} in order to obtain the desired orientation. □

Now we show that, in G, every vertex of small degree has a neighbor of higher degree.

Proposition 6. *Let u be a vertex of G. If $d(u) \leq 7$, then there exists $v \in N(u)$ such that $d(v) > d(u)$.*

Proof. Suppose for a contradiction that $d(u) \leq 7$ and all vertices in $N(u)$ have degree at most $d(u)$. Let D be a proper 7-orientation of $G - u$. For each $v \in N_G(u)$, since $d_{G-u}(v) = d_G(v) - 1 \leq d_G(u) - 1$, we know that $d^-_D(v) < d_G(u)$. Therefore, because $d_G(u) \leq 7$, one can extend D by orienting every edge incident to u in G towards u to obtain a proper 7-orientation of G, a contradiction. □

Proposition 7. *Every leaf block of G is either a 2-block or a 3-block.*

Proof. Observe that, for any leaf block with at least four vertices, there must be at least one vertex of degree 2 whose neighbors also have degree 2, contradicting Proposition 6. □

Proposition 7 implies that a leaf block is either a 1-*path* (i.e. a path of length 1) or a *triangle* (i.e. a cycle of length 3). In Figure 2, we present every possible proper orientation of a leaf block.

Proposition 8. *Every vertex of G is contained in at most one leaf 2-block.*

Proof. By contradiction, suppose that it is not the case and let $\langle u, v\rangle, \langle u, w\rangle$ be two leaf 2-blocks containing u. Let D be a proper 7-orientation of $G - w$. If $d^-_D(u) \neq 1$, orienting uw towards w extends D into a proper 7-orientation of G, a contradiction. Hence $d^-_D(u) = 1$. Since D is proper, the edge $uv \in E(G)$ must

(a) A_1 (b) A_2 (c) T_1 (d) T_2 (e) T_3

Figure 2: Leaf blocks and their possible proper orientations.

be the only one oriented towards u in D. Therefore all neighbors of u distinct from v and w have in-degree greater than 1 in D. Reverting the orientation of uv in D and orienting uw towards w, we obtain a 7-orientation of G, which is proper because the in-degree of u is 0, hence different from the in-degree of all of its neighbors. This is a contradiction. $\square$

Proposition 9. *Every twig block is a* 2*-block or a* 3*-block.*

Proof. Let B be a twig block of order q at least 4, say $B = \langle u_1, \dots, u_q \rangle$ with u_1 the root of B.

Claim 9.1. $d(u_i) \neq 3$*, for every* $i \in \{2, \dots, q\}$.

Subproof. By contradiction, suppose that there exists a vertex $u_i \in \{u_2, \dots, u_q\}$ of degree 3 in G. Note that u_i is contained in the block B and in a leaf 2-block, say $\langle u_i, v \rangle$.

First suppose that $i \notin \{2, q\}$ and let $G' = G - \{u_i, v\}$. By the minimality of G, there exists a proper 7-orientation D of G'. If $\{d^-_D(u_{i-1}), d^-_D(u_{i+1})\} \neq \{2, 3\}$, then one could extend D to a proper 7-orientation of G by orienting u_iu_{i-1} and u_iu_{i+1} towards u_i and choosing the orientation of u_iv according to the in-degrees of u_{i-1} and u_{i+1} in D, a contradiction. Hence, without loss of generality, consider that $d^-_D(u_{i-1}) = 2$ and $d^-_D(u_{i+1}) = 3$.

Let us extend D by orienting all the arcs incident to u_i away from this vertex. The resulting orientation D' is not yet proper but we shall prove how to change it into a proper 7-orientation of G. Problems could only appear in edges incident to u_{i-1} or u_{i+1} which had in-degree 3 and 4 respectively in D. Observe that these two vertices have degree more than 2 and thus belong to some other blocks which must be leaf blocks since u_1 is the root of B. One can reorient the leaf blocks containing u_{i+1} using the orientations of Figure 2 so that the in-degree of u_{i+1} becomes 3 again. Similarly, if $d(u_{i-1}) = 4$, one can reorient the leaf blocks containing u_{i-1} so that the in-degree of u_{i-1} is in $\{3, 4\} \setminus \{d^-_D(u_{i-2})\}$, and if $d(u_{i-1}) = 3$ (that is u_{i-1} is in a unique leaf 2-block), one can reorient the leaf block containing u_{i-1} so that the in-degree of u_{i-1} becomes 2 again. The resulting orientation is then a proper 7-orientation of G, a contradiction.

Suppose now that $i \in \{2, q\}$. Without loss of generality, we may assume that $i = 2$. Let G' be the connected component of $G - u_3$ containing u_2. Let D' be a proper 7-orientation of G'. Clearly, $d^-_{D'}(u_2) \leq 2$. By the previous paragraphs and because $q \geq 4$, we know that $d_G(u_3) \neq 3$. If $d_G(u_3) > 3$, we can obtain a proper orientation of G by orienting edges u_2u_3 and u_3u_4 towards u_3 and orienting the leaf blocks containing u_3 in such a way that $d^-(u_3) \in \{3,4\} \setminus d^-_D(u_4)$; this is a contradiction. Consequently, $d(u_3) = 2$, and we can suppose that $2 \in \{d^-_D(u_2), d^-_D(u_4)\}$, as otherwise we get a contradiction by adding (u_2, u_3) and (u_4, u_3).

First, suppose that $d^-_D(u_2) \neq 2$, in which case one can verify that we can suppose that $d^-_D(u_2) = 0$. If $d(u_4) > 3$, we reorient the leaf blocks and u_3u_4 so that u_4 has in-degree in $\{3,4\} \setminus \{d^-_D(u_5)\}$, then we let u_3 have in-degree 1 or 2, depending on the orientation of u_3u_4. This gives us a contradiction, and, because $d^-_D(u_4) = 2$, we get that $d(u_4) = 3$ and, by the previous paragraphs, that $q = 4$. Let $v' \in N(u_4) \setminus B$. We get a contradiction by reversing (v', u_4) and adding (u_2, u_3) and (u_3, u_4).

Finally, suppose that $d^-_D(u_2) = 2$. Then we can also suppose that $d^-_D(u_4) = 1$ as otherwise we can reverse (v, u_2), orient u_2u_3 towards u_2, and orient u_3u_4 towards u_3 to obtain a proper 7-orientation of G. By similar arguments, if $d(u_4) > 3$, then we can change its in-degree to some $c \in \{3,4\} \setminus \{d^-_D(u_5)\}$; hence $d(u_4) \in \{2,3\}$ and we analyse the cases below:

- $d(u_4) = 3$: let $v' \in N(u_4) \setminus B$. Because $d^-(u_4) = 1$, we know that $(v', u_4), (u_4, u_1) \in D$. Reverse (v, u_2) and (v', u_4), and add (u_3, u_2) and (u_4, u_3) to obtain a contradiction;

- $d(u_4) = 2$: if $q = 4$, because $d^-_D(u_2) = 2$ we know that $d^-_D(u_1) \neq 2$. Reverse (v, u_2) and add (u_3, u_2) and (u_3, u_4) to obtain a contradiction. Otherwise, by similar arguments we can suppose that $d^-_D(u_5) = 2$. Suppose that $d(u_5) > 3$ and reorient the leaf blocks containing u_5 and u_4u_5 so that u_5 has in-degree in $\{3,4\} \setminus \{d^-_D(u_6)\}$. After this, u_4 has in-degree either 0 or 1, in which case we reverse (v, u_2), add (u_3, u_2) and either (u_4, u_3) or (u_3, u_4), depending on u_4. Finally, we can suppose that $d(u_5) = 3$ and $q = 5$. Let $v' \in N(u_5) \setminus B$. Reverse (v, u_2) and (v', u_5), and add (u_3, u_2), (u_4, u_5) and (u_3, u_4) to get a contradiction.

$\diamondsuit$

Now we return to the proof of the proposition. By the minimality of G, there is a proper 7-orientation D of $G\langle u_1 \rangle$.

We shall extend D into a proper 7-orientation of G, which gives us the desired contradiction. We first add $(u_1, u_2), (u_1, u_q)$. We then distinguish some cases according to $d^-_D(u_1)$.

Assume first $d^-_D(u_1) \notin \{2,4\}$. Add $(u_3, u_2), (u_{q-1}, u_q)$ and orient the path $(u_3, \ldots, u_{q-1})$ according to Lemma 5. So far the vertices $u_2, \ldots, u_q$ have in-degree 0, 1, or 2 in B. For each $i \in \{2, \ldots, q\}$, if u_i is contained in some leaf block, then by Claim 9.1 $d(u_i) \geq 4$. Thus, by Proposition 8, u_i is in at least one leaf 3-block. If u_i has in-degree 0 in B, then we orient all the leaf blocks containing u_i with A_1 or T_1, so that u_i still has in-degree 0. If u_i has in-degree 1 (resp. 2) in B, we orient one leaf 3-block according to

T_3 and all other blocks according to A_1 and T_1, so that its in-degree is 3 (resp. 4). It is now a simple matter to check that the obtained orientation is a proper 7-orientation of G.

Assume now $d_D^-(u_1) \in \{2,4\}$. If $q = 4$, add (u_2, u_3) and (u_4, u_3), and one can verify that we can get a contradiction again by orienting the leaf blocks containing vertices in B in the same way as above. So, suppose that $q \geq 6$. Add (u_2, u_3), (u_4, u_3), (u_q, u_{q-1}), and (u_{q-2}, u_{q-1}). Furthermore, if $q = 7$ then add (u_4, u_5), and if $q > 7$ apply Lemma 5 to orient the path $(u_4, \ldots, u_{q-2})$. We then orient the leaf blocks containing vertices in B in the same way as above to get a contradiction.

Therefore, we can consider $q = 5$. Add the arcs (u_1, u_2), (u_1, u_5), (u_3, u_4), and (u_5, u_4) to D.

If $d(u_2) > 2$, then u_2 is in a leaf 3-block. Add (u_3, u_2), and orient one leaf 3-block containing u_2 with T_2 and the other leaf blocks with A_1 or T_1 so that u_2 has in-degree 3. For $j \in \{3,4,5\}$, if u_j is contained in some leaf block, orient its leaf blocks so that the in-degree of u_j increases by 2 (using one T_3 and possibly some A_1 and T_1). It is simple matter to check that it gives a proper 7-orientation of G. By symmetry, we get the result if $d(u_5) = 2$.

Finally, consider $d(u_2) = d(u_5) = 2$, and since B is not a leaf block, we can suppose, without loss of generality, that $d(u_3) > 2$. In this case, add (u_2, u_3), (u_3, u_4) and (u_5, u_4), orient the leaf block(s) containing u_3 so that its in-degree is 3 and, if $d(u_4) > 2$, orient the leaf block(s) containing u_4 so that its in-degree is 4.

□

Proposition 10. *Let B be a twig block with root u_1.*

(a) If $B = \langle u_1, u_2 \rangle$, then either $d(u_2) = 2$ or u_2 belongs exactly to B and to a leaf 3-block.

(b) If $B = \langle u_1, u_2, u_3 \rangle$, then, for each $j \in \{2,3\}$, u_j belongs exactly to B and either a leaf 2-block or a leaf 3-block.

Proof. (a) Assume that $d(u_2) > 2$. Let D be a proper 7-orientation of $G\langle u_1 \rangle$. We can suppose that $d(u_2) = 3$, as otherwise we extend D to a proper 7-orientation of G by orienting u_1u_2 towards u_2 and orienting the leaf blocks with root u_2 in such a way that its in-degree belongs to $\{3,4\} \setminus \{d_D^-(u_1)\}$. Consequently, by Proposition 7 and Proposition 8, we obtain that u_2 is contained exactly in B and in a leaf 3-block.

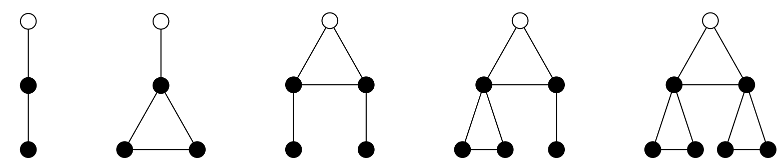
(b) Suppose first that one vertex of $\{u_2, u_3\}$, say u_3, is in no leaf block, since B is a twig block, so $d(u_2) \geq 3$.

Suppose $d(u_2) > 3$ and let D be a proper 7-orientation of $G\langle u_1 \rangle$. One can orient the edges u_1u_2 and u_1u_3 from u_1 to its neighbors and then orient the leaf block(s) containing u_2 and the edge u_2u_3 in such a way that the in-degree of the pair (u_2, u_3) is $(3, 2)$, in case $d^-_D(u_1) \notin \{2, 3\}$, or $(4, 1)$, otherwise. This results in a proper 7-orientation of G, a contradiction.

If $d(u_2) = 3$, then let D be a proper 7-orientation of $G - v$, where v is the neighbor of u_2 not in B. Since u_2 and u_3 are symmetric in $G-v$, we can suppose that $d^-_D(u_2) \neq 1$, in which case we can extend D into a proper 7-orientation of G by orienting u_2v towards v. This is a contradiction.

Suppose now that $d(u_2) > 2$, and $d(u_3) > 2$. If $d(u_2) \geq 5$, let G' be the component of $G - u_2$ containing u_1. Let D be a proper 7-orientation of G'. One could then extend D to a proper 7-orientation of G by orienting the edges u_1u_2 and u_2u_3 towards u_2 and orienting the leaf blocks containing u_2 in such a way that its in-degree belongs to $\{3, 4, 5\} \setminus \{d^-_D(u_1), d^-_D(u_3)\}$. By symmetry, we get a contradiction in the same way if $d(u_3) \geq 5$. Therefore $d(u_2) \leq 4$ and $d(u_3) \leq 4$. Then, the proposition follows by Proposition 7 and by Proposition 8. $\square$

The 2-*path*, the *kite*, the *bull*, the *elk*, and the *moose* are the rooted graphs depicted in Figure 3 where the root is the white vertex.



(a) 2-path (b) kite (c) bull (d) elk (e) moose

Figure 3: The five possible twig subgraphs.

Propositions 8, 9, and 10 imply directly the following.

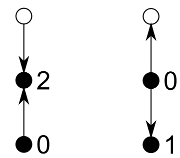
Corollary 11. *Every twig subgraph in G is either a 2-path, or a kite, or a bull, or an elk, or a moose.*

In the following we will very often use this corollary without referring explicitly to it.

All the possible (partial) proper orientations of the twig subgraphs are depicted in Figures 4 to 8. In these figures, the notation $i - j$ means that the corresponding vertex can have any in-degree in this range, depending on the orientation given to the non-oriented edges.

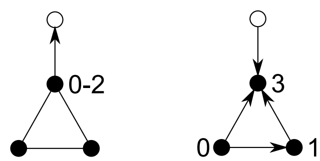
Proposition 12. *Let B be a bough block with root u. Every vertex v in $V(B) \setminus \{u\}$ with degree at least 3 is the root of a twig subgraph or a leaf block that is neither a kite nor a moose.*

Proof. Let v be a vertex in $V(B) \setminus \{u\}$ with degree at least 3. It must be the root of at least one twig subgraph or leaf block. Suppose the contrary that v is only root of kites and moose. Let S be the set of vertices that belong to such



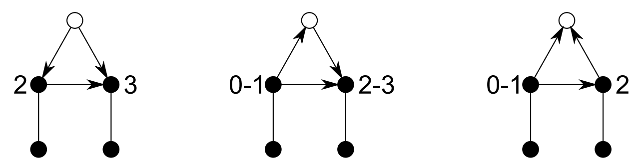
(a) P_1 (b) P_2

Figure 4: Proper orientations of the 2-path.



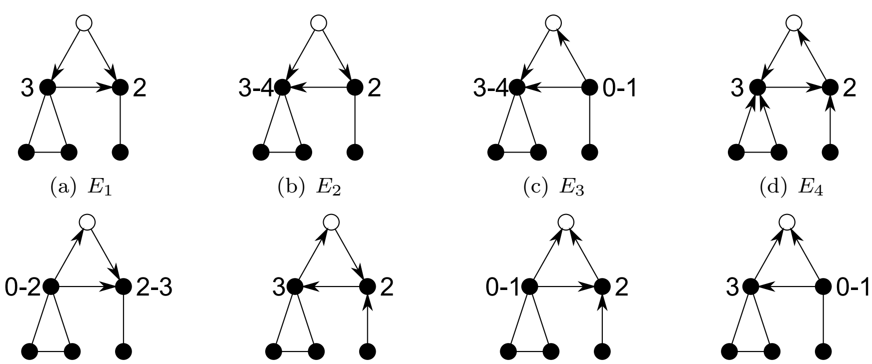
(a) K_1 (b) K_2

Figure 5: Proper orientations of the kite.



(a) B_1 (b) B_2 (c) B_3

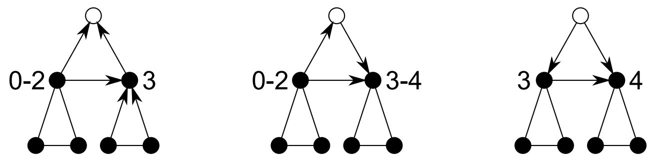
Figure 6: Proper orientations of the bull.



(a) E_1 (b) E_2 (c) E_3 (d) E_4

(e) E_5 (f) E_6 (g) E_7 (h) E_8

Figure 7: Proper orientations of the elk.



(a) M_1 (b) M_2 (c) M_3

Figure 8: Proper orientations of the moose.

kites and moose rooted at v. Let D be a proper 7-orientation of the subgraph of G induced by $(V(G) \setminus S) \cup \{v\}$. Observe that $d_D^-(v) \leq 2$. Thus, one could extend D to G by orienting the kites and moose rooted at v according to K_2 or M_3. $\square$

Proposition 13. *Let $B = \langle u_1, \ldots, u_q \rangle$ be a bough block with root u_1. For all $i \in \{2, \ldots, q\}$, $d(u_i) \leq 4$.*

Proof. Let $i \in \{2, \ldots, q\}$. Let G' be the connected component containing u_1 in $G - u_i$. By the minimality of G, G' admits a proper 7-orientation D. Set $F = \{d_D^-(u_{i-1}), d_D^-(u_{i+1})\}$. Add the arcs (u_{i-1}, u_i) and (u_{i+1}, u_i).

If $d(u_i) \geq 7$, then one can properly orient the twig subgraphs and leaf blocks with root u_i in such a way that u_i has in-degree in $\{5, 6, 7\} \setminus F$. Observe that all other vertices of those graphs have in-degree at most 4, so we obtain a proper 7-orientation of G, a contradiction.

If $d(u_i) = 6$, by Proposition 12, it is contained in at most one moose. Therefore, one can orient the twig and leaf subgraphs containing u_i so that u_i has in-degree in $\{4, 5, 6\} \setminus F$, taking care to use M_1 for the possible moose. Observe that every other possible twig can avoid a 4 from appearing in $N(u_i)$. Hence, we have a proper 7-orientation of G, a contradiction.

Thus, we can suppose that $d(u_i) \leq 5$, for all $i \in \{2, \ldots, q\}$.

Assume now for a contradiction that there is some $i \in \{2, \ldots, q\}$ such that $d(u_i) = 5$.

If $q = 2$, then $|F| = 1$ and one can extend D to a proper 7-orientation of G by orienting the twig and leaf blocks containing u_i so that the in-degree of u_i belongs to $\{4, 5\} \setminus F$. This is a contradiction so $q \geq 3$.

Observe that if $\{4, 5\} \neq F$, then one can extend D to G by orienting the twig and leaf blocks containing u_i in such a way that its in-degree belong to $\{4, 5\} \setminus F$. Consequently, we can assume that $F = \{4, 5\}$. But $d(u_j) \geq d_D^-(u_j) + 1$. So one vertex in $\{u_{i-1}, u_{i+1}\}$ is u_1. Free to relabel the vertices in the other sense around B, we may assume that $i = 2$. Hence $d_D^-(u_1) = 5$ and $d_D^-(u_3) = 4$. So $d(u_3) = 5$. Applying the same reasoning to u_3, we obtain that $q = 3$.

Claim 13.1. *There is a proper 7-orientation D' of G' such that $d_{D'}^-(u_3) \in \{2, 3\}$.*

Subproof. The idea is to start form D and to reorient the edges of the leaf blocks and twig subgraphs with root u_3. Observe that in D all the edges incident to u_3 are directed towards u_3. In particular (u_1, u_3) is an arc of D.

By Propositions 7, 9 and 10, u_3 is the root of:

1. two subgraphs, H_1 and H_2, with H_1 being a triangle, a bull, an elk or a moose, and H_2 being a 1-path, a 2-path, or a kite; or
2. three subgraphs, H_1, H_2 and H_3, each of them being a 1-path, a 2-path, or a kite.

If Case 1 occurs, then we are in one of the following subcases.

1.1. H_1 is a moose. Orient it using M_3 and H_2 using A_2, P_2 or K_1 (with the in degree of its neighbor 0). This yields the desired proper orientation D' with $d^-_{D'}(u_3) = 2$.

1.2. H_1 is an elk or a bull. If H_2 is a 1-path or 2-path, then orient H_1 with E_7 or B_3 and H_2 with A_1 or P_1 to obtain the desired orientation D' with $d^-_{D'}(u_3) = 3$. If not, then H_2 is a kite. Orient H_1 with E_3 or B_2 (with the neighbor of u_3 having in degree different from 2) and H_2 with K_2 to obtain the desired orientation D' with $d^-_{D'}(u_3) = 2$.

1.3 H_1 is a triangle. Orient H_1 with T_2 and H_2 with A_2, P_2 or K_1 to obtain the desired orientation D' with $d^-_{D'}(u_3) = 3$.

If Case 2 occurs, without loss of generality, we are in one of the following subcases.

2.1 H_1 and H_2 are kites. Orient H_1 and H_2 using K_2 and H_3 using A_2, P_2 or K_1, to obtain the desired orientation D' with $d^-_{D'}(u_3) = 2$.

2.2 H_1 is a kite or a 1-path or a 2-path, and H_2 and H_3 are 1-path or a 2-path. Orient H_1 using K_1 or A_2 or P_2, H_2 using A_2 or P_2, and H_3 using A_1 or P_1, to obtain the desired orientation D' with $d^-_{D'}(u_3) = 3$.

$\Diamond$

Now apply the above reasoning with the orientation D' given by Claim 13.1: we have $F \neq \{4, 5\}$ because $d^-_{D'}(u_3) \in \{2, 3\}$. Therefore, we obtain a proper 7-orientation of G, a contradiction. $\square$

Proposition 6 implies the following.

Proposition 14. *Let u be a vertex in G.*

(a) if u is the root of a kite or a bull, then $d(u) \geq 4$;

(b) if u is the root of an elk or a moose, then $d(u) \geq 5$.

Proposition 15. *Every bough block is a 3-block.*

Proof. Let $B = \langle u_1, \dots, u_q \rangle$ be a block with root u_1.

Assume first that $q = 2$. Let D be a proper 7-orientation of $G\langle u_1 \rangle$. By Proposition 13, we know that $d(u_2) \leq 4$. If $d(u_2) = 4$, we can orient the remaining edges in such a way that u_2 has in-degree in $\{3, 4\} \setminus \{d^-_D(u_1)\}$ taking care that all kites are oriented using K_1. This is possible because u_2 is the root of at most two kites thanks to Proposition 12. This yields a proper 7-orientation of G, a contradiction.

Henceforth, since B is a bough block, u_2 is the root of a twig subgraph H_1. In particular, $d(u_2) = 3$, and by Proposition 14, H_1 is a 2-path, say (u_2, x, x'). Vertex u_2 must also be the root of another subgraph H_2 that is either a 2-path (u_2, y, y') or a 1-path (u, y). Add the arc (u_1, u_2). If $d^-_D(u_1) \neq 3$, one can orient H_1 and H_2 using P_2 and A_2 so that u_2 get in-degree 3. This yields a proper 7-orientation of G, a contradiction. Assume $d^-_D(u_1) = 3$. If H_2 is a 2-path, then orient H_1 and H_2 using P_1 so that u_2 get in-degree 1. If H_2 is a 1-path, then orient H_1 using P_2 and H_2 using A_1 so that u_2 get in-degree 2. In both cases, it results in a proper 7-orientation of G, a contradiction.

Now, suppose that $q \geq 4$. Note that Propositions 6 and 13 imply $d(u_3) \leq 3$, and that Proposition 14 implies that u_3 is not root of a kite. So, either $d(u_3) = 2$ or u_3 is the root of a 1-path or a 2-path.

Suppose first that $d(u_2) = 4$. Let D be a proper orientation of $G\langle u_2 \rangle - u_2$. Because $d(u_3) \leq 3$, we get that $d^-_D(u_3) \leq 2$. Add the arcs (u_1, u_2) and (u_3, u_2). By Proposition 12, u_2 is neither the root of a moose nor of two kites. Therefore, one can orient the twig subgraphs and leaf blocks with root u_2 so that its in-degree belongs to $\{3, 4\} \setminus \{d^-_D(u_1)\}$. This results in a proper 7-orientation of G, a contradiction.

Similarly, we get a contradiction if $d(u_{q-1}) = 4$, so we can assume that: $(\star)$ $d(u_i) \leq 3$, for all $i \in \{2, \dots, q\}$.

Now by Proposition 14-(a), if $d(u_i) = 3$ for some $i \in \{2, \dots, q\}$, it is the root of a 1-path or a 2-path. Consequently, by $(\star)$, for all $i \in \{3, \dots, q-1\}$, it has no neighbor of degree more than 3. Thus, by Proposition 6, we get $d(u_i) = 2$, for every $i \in \{3, \dots, q-1\}$, and $q \leq 5$, for otherwise u_4 has degree 2 and no neighbor of degree more than 2.

Since B is a bough block and not a twig block, one of its vertices distinct from the root u_1 must be the root of a twig subgraph. Necessarily, it must be u_2 or u_q as all other vertices have degree 2. By symmetry, we may assume that it is u_2. Furthermore, since $d(u_2) = 3$, by Proposition 14-(a), u_2 is necessarily the root of a 2-path, say (u_2, x, x').

Let D be a proper 7-orientation of $G\langle u_1 \rangle$. Orient the edges u_1u_2, u_1u_q and u_2u_3 towards u_2, u_q and u_3, respectively. We now describe how to extend this orientation in a proper 7-orientation of G, yielding the contradiction. We distinguish two cases depending on whether $q = 4$ or $q = 5$.

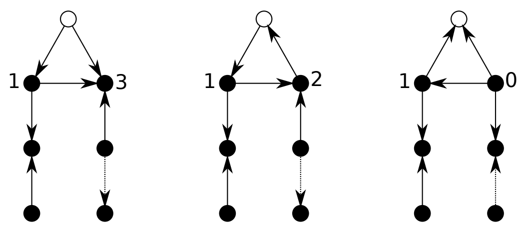
- $q = 4$. Assume first $d(u_4) = 2$. If $d^-_D(u_1) \neq 2$, add (u_3, u_4), (x, u_2) and (x, x'); otherwise, add their reverses. So suppose that $d(u_4) = 3$. Then u_4 is the root of either a 1-path (u_4, y) or a 2-path (u_4, y, y') by Proposition 14. If $d^-_D(u_1) \neq 3$, then D can be extended to G by reversing

u_2u_3 and orienting the remaining edges so that the in-degrees of u_2 and u_4 will be 3. If $d^-_D(u_1) = 3$. Add (u_4, y). If u_4 is the root of a 1-path, add (u_3, u_4), (x, u_2) and (x, x'). Otherwise,u_4 is the root of a 2-path : add (u_4, u_3), (u_2, x), (x', x), and (y', y).

- $q = 5$. By Proposition 6, we have $d(u_5) = 3$. So u_5 is the root of either a 1-path (u_5, y) or a 2-path (u_5, y, y') by Proposition 14. If $d^-_D(u_1) \neq 3$, reverse u_2u_3 and orient properly the remaining edges in a way that the in-degrees of u_2 and u_5 is 3. If $d^-_D(u_1) = 3$, first add (u_2, x), (x', x) and (u_4, u_3) to D. If u_5 is the root of a 1-path, then add (u_5, y) and (u_4, u_5); otherwise, u_5 is the root of a 2-path : add (y, u_5), (u_5, u_4) and (y, y').

□

A *reindeer* is the graph depicted in Figure 9, where the root is the white vertex. It also depicts all possible orientations of the reindeer.



(a) R_1 (b) R_2 (c) R_3

Figure 9: The reindeer and its possible orientations. The dashed edge may or may not exist.

Proposition 16. *Every bough subgraph is a reindeer.*

Proof. Let H be a bough subgraph rooted at u_1. It contains a bough block B. By Proposition 15, B is a 3-block, say $B = \langle u_1, u_2, u_3 \rangle$. By Proposition 13, $d(u_2) \leq 4$ and $d(u_3) \leq 4$.

Let G' be the connected component of $G - u_2$ containing u_1. Let D be a proper 7-orientation of G'.

Assume $d(u_2) = 4$. By Proposition 14, u is the root of no moose nor elk, and by Proposition 12, it is the root of at most one kite. If $\{d^-_D(u_1), d^-_D(u_3)\} \neq \{3, 4\}$, then adding (u_1, u_2) and (u_3, u_2) and using appropriate orientations of the twig subgraphs and leaf blocks with root u_2, one can get an orientation of D such that $d^-(u_2) \in \{3, 4\} \setminus \{d^-_D(u_1), d^-_D(u_3)\}$. This is a proper 7-orientation of D, a contradiction. Consequently, $\{d^-_D(u_1), d^-_D(u_3)\} = \{3, 4\}$, and so $d^-_D(u_3) = d_G(u_3) - 1 = 3$. Let x be a neighbor of u_3 not in B and let H be the twig subgraph or leaf block with root u_3 containing x. By Proposition 12, one can choose x so that H is not in a kite. Add (u_2, u_3) and use A_1, T_2, P_1, or B_2 to reverse (x, u_3). If u_2 is not the root of two 2-paths, we can orient the twig subgraphs and leaf blocks with root u_2 so that its in-degree becomes 2 by using orientations A, T_2, P_2, K or B_2. If u_2 is the root of two 2-paths, we can orient these 2-paths using P_2 so that u_2 gets in-degree 1. In both cases, we obtain a proper 7-orientation of D, a contradiction.

Similarly, we get a contradiction if $d(u_3) = 4$. Therefore $d(u_2) \leq 3$ and $d(u_3) \leq 3$. Since B is a bough block, u_2 or u_3 must be the root of a twig subgraph. Without loss of generality, we may assume that u_2 is. By Proposition 14, u_2 must be the root of a 2-path, say (u_2, x, x').

Assume $d(u_3) = 2$. If $d^-(u_1) \notin \{1, 2\}$, add $(u_2, x), (u_2, u_3), (x', x)$, and if $(u_3, u_1) \in D$, reverse it and add (u_2, u_1); otherwise, add (u_1, u_2). And if $d^-(u_1) \in \{1, 2\}$, add (u_1, u_2), (u_3, u_2), (x, u_2) and (x, x'). In both cases, it results in a proper 7-orientation of D, a contradiction.

Hence $d(u_3) = 3$, which by Proposition 12 implies that u_3 is the root of either a 2-path or a 1-path. Therefore H is a reindeer. $\square$

We can finally prove the main result of this paper.

Proof of Theorem 4. If G has no branch blocks, then there exists a vertex u such that G is the union of bough subgraphs, twig subgraphs and leaf blocks with root u. In this case, one may obtain a proper 4-orientation of G by orienting all bough subgraphs, twig subgraphs and leaf blocks so that the in-degree of u is 0.

Thus, G contains a branch block B. It must contain a vertex u which is the root of a bough subgraph R. By Proposition 16, R is a reindeer, and by Proposition 6, we have $d(u) \geq 4$. Denote by Q the subgraph rooted at u containing exactly all the bough, twig and leaf blocks rooted at u.

Let H be the component of $G - u$ that contains $B - u$; then u has at most 2 neighbors in H. By minimality of G, H has a proper 7-orientation D. Let F be the set of in-degrees of neighbors of u in H. Orient the edges of H incident to u towards u.

If $d(u) \geq 7$, we can orient $G\langle u\rangle$ in such a way that u has in-degree in $\{5, 6, 7\} \setminus F$ and no vertex in Q has in-degree more than 4. This gives a proper 7-orientation of G, a contradiction.

Assume $d(u) = 6$. Let α be an integer in $\{4, 5, 6\} \setminus F$. We can orient Q in such a way that u has in-degree α and no vertex of $Q - u$ has in-degree α. This is possible because no vertex has in-degree 5 in the orientations depicted in Figures 2, 4–8 and 9 and u is in at most two moose, so if $\alpha = 4$, we can orient the moose first using M_1 or M_2. This gives a proper 7-orientation of G, a contradiction.

Assume $d(u) = 4$. If u has two neighbors in H, then $Q = R$. Let α be an integer in $\{2, 3, 4\} \setminus F$. If $\alpha = 2$, then orient R with R_1; if $\alpha = 3$, then orient R with R_2; if $\alpha = 4$, then orient R with R_3. In each case, this yields a proper 7-orientation of G, a contradiction. If u has a unique neighbor in H, then Q is the union of R and either a 1-path, or a 2-path, or a kite. Orient that subgraph using A_2, P_2 or K_1. Now, since $|F| = 1$, we can orient R using R_2 or R_3 so that the in-degree of u in $\{3, 4\} \setminus F$. This yields a proper 7-orientation of G, a contradiction.

Finally assume $d(u) = 5$. If $F \neq \{4, 5\}$, we can orient the edges of Q so that the in-degree of u is some $\alpha \in \{4, 5\} \setminus F$, and no vertex of $Q - u$ has in-degree

α. If $\alpha = 4$, this is possible because u is in at most one moose, and we can start orienting the moose with M_2. This yields a proper 7-orientation of G, a contradiction. If $F = \{4, 5\}$, then Q is the union of R and either a 1-path or a 2-path or a kite. In the first two cases, orient the 1-path or 2-path by using A_1 or P_1, and R with R_2, so that vertex u has in-degree 3. In the latter case, orient the kite with K_2 and R with R_1, so that vertex u has in-degree 2. In both cases, we obtain a proper 7-orientation of G, a contradiction. □

3. A tight example

Recall that a *block* graph is a graph such that each block is a clique. In the sequel, we find a tight example for Theorem 4. As a drawback, we obtain another tight example for Theorem 1 different to the one the authors in [1] propose and an example of a planar graph whose proper orientation must be at least 10.

Theorem 17. *Let k be a positive integer. There exists a block graph $G(k)$ such that $\omega(G) = k$ and $\overrightarrow{\chi}(G) \geq 3k - 2$.*

Let G be a connected graph, and K be a clique in G. We say that K is a *pending clique of* G if there exists $u \in K$ such that there are no edges between $K - u$ and $V(G) - u$. We say that u is the *root of* K.

Lemma 18. *Let G be a connected graph, K be a pending clique of G with size k and root u, and D be a proper orientation of G. If u has an in-neighbor in $V(G) \setminus K$, then $d^-_D(u) \geq k$.*

Proof. By contradiction, suppose that u has an in-neighbor in $V(G) \setminus K$ and that $d^-_D(u) = d \in \{1, \ldots, k-1\}$. Because $d \in \{1, \ldots, k-1\}$, and $d(v) = k-1$ for every $v \in K \setminus \{u\}$, we necessarily have that $\{d^-_D(v) \mid v \in K\} = \{0, \ldots, k-1\}$. Consequently, there exist d vertices $k_{i_0}, \ldots, k_{i_{d-1}}$ in K such that $d^-_D(k_{i_j}) = j$, for every $j \in \{0, \ldots, d-1\}$. Define $k_{i_d} = u$, similarly. Observe that all edges $k_{i_j}u$ must be oriented towards u, since $k_{i_j}k_{i_\ell}$ must be oriented towards k_{i_ℓ}, whenever $0 \leq j < \ell \leq d$. This is a contradiction, because u has another in-neighbor that does not belong to K and thus $d^-_D(u) \geq d + 1$. □

A *k-chandelier* is the graph obtained from a k-clique $K = \{v_0, \ldots, v_{k-1}\}$ by adding $k-1$ pending k-cliques in the vertices $v_1, \ldots, v_{k-1}$. We say v_0 is the *root* of the k-chandelier and K is its *base*.

Lemma 19. *Let G be a k-chandelier with root v_0 and base $K = \{v_0, \ldots, v_{k-1}\}$. If D is a proper orientation of G such that $(v_0, v_i) \in A(D)$ for every $i \in \{1, \ldots, k-1\}$, then $d^-_D(v_0) \notin \{k, \ldots, 2k-2\}$.*

Proof. Consider any $i \in \{1, \ldots, k-1\}$. Since $(v_0, v_i) \in A(D)$, Lemma 18 yields $d^-_D(v_i) \geq k$. In addition $d^-_D(v_i) \leq 2k-2$, because $d(v_i) = 2k-2$. Therefore, we must have $\{d^-_D(v_i) \mid v_i \in K - v_0\} = \{k, \ldots, 2k-2\}$ and the lemma follows, because D is a proper orientation. □

Proof of Theorem 17. Let $G(k)$ be the graph obtained as follows: we start with a k-clique $K = \{v_1, \ldots, v_k\}$ and then we add $2k-1$ pending k-cliques $C_{i,j}$ on each v_i, for every $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, 2k-1\}$. Define

$$B = \bigcup_{i=1}^{k} \bigcup_{j=1}^{2k-1} C_{i,j}.$$

Note that at this point B contains all vertices we have added to $G(k)$ so far. Then, for each $u \in B$, we add $3k-2$ copies of a k-chandelier and 2 pending k-cliques, all of them with root u. This finishes the construction of $G(k)$.

Suppose for a contradiction that there exists a proper orientation D of $G(k)$ such that $\Delta^-(D) \leq 3k-3$.

We claim that, for every $u \in B$, we have that $d_D^-(u) \notin \{1, \ldots, 2k-2\}$. Indeed, suppose that $d_D^-(u) \neq 0$ and thus that u has an in-neighbor v. One of the two k-cliques pending in u does not contain v, so by Lemma 18, $d_D^-(u) \geq k$. Now recall that $d_D^-(u) \leq \Delta^-(D) \leq 3k-3$. Therefore, u has no in-neighbors in at least one of the $3k-2$ k-chandeliers with root u. Hence, by Lemma 19, $d_D^-(u) \notin \{k, \ldots, 2k-2\}$. This proves our claim.

Therefore the in-degrees of the vertices of B are in $\{0, 2k-1, \ldots, 3k-3\}$. There are exactly k values in this set, so each k-clique in B must have exactly one vertex of each in-degree in this set. In particular, each of these cliques of B must contain a vertex of in-degree 0. Consider the vertex $v_i \in K$ such that $d_D^-(v_i) = 2k-1$. Let $u_0 \in K$ be such that $d_D^-(u_0) = 0$, and, for each $j \in \{1, \ldots, 2k-1\}$, let $u_j \in C_{i,j}$ be such that $d_D^-(u_j) = 0$. Since all edges $u_j v_i$ are oriented towards v_i, we have that $d_D^-(v_i) \geq 2k$, a contradiction. □

One may see that Theorem 17 provides a tight example for Theorem 1 when $k = 2$ and a tight example for Theorem 4 for $k = 3$.

Corollary 20. *There exist cacti G such that $\overrightarrow{\chi}(G) \geq 7$.*

Since every block graph G with $\omega(G) = 4$ is planar, we also have the following corollary:

Corollary 21. *There exist planar graphs G such that $\overrightarrow{\chi}(G) \geq 10$.*

4. Further Research

4.1. Proper-orientation number of planar graphs

We believe that Problem 3 must be answered in the affirmative: outerplanar graphs have proper-orientation number bounded by a constant c. If such a c exists, then $c \geq 7$, since cacti (and in particular, the one described in Section 3) are outerplanar. A first step would be to established the result for 2-connected outerplanar graphs. We actually believe that in this case this constant should be smaller than 7 and that it should not be much greater than 3. One can easily attain 3 as a lower bound using the following lemma.

Lemma 22 ([1]). *Let k be a positive integer, and let G be a graph containing a clique K of size $k+1$. In any proper k-orientation of G, all edges between $V(K)$ and $V(G) \setminus V(K)$ are oriented from $V(K)$ to $V(G) \setminus V(K)$.*

Proposition 23. *There exists a 2-connected outerplanar graph G such that $\overrightarrow{\chi}(G) = 3$.*

Proof. Let G be the graph on six vertices defined by $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{v_1v_2, v_2v_3, v_1v_3, v_4v_5, v_5v_6, v_4v_6, v_1v_4, v_2v_5\}$. Suppose by way of contradiction that G has a proper 2-orientation D. Observe that the sets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ are cliques in G. Thus Lemma 22 implies that the edges v_1v_4 and v_2v_5 must be oriented in both ways, a contradiction. $\square$

To the more general case of planar graphs, similarly, it would be interesting to find a constant c', if it exists, satisfying $\overrightarrow{\chi}(G) \leq c'$, for every planar graph G. We provided in Section 3 a planar graph whose proper orientation number is 10 and thus $c' \geq 10$.

4.2. $\overrightarrow{\chi}$-bounded families of graphs

Gyárfás [5] introduced the concept of χ-*bounded* graph classes. A class of graph $\mathcal{G}$ is said to be χ-*bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Similarly, one can define $\overrightarrow{\chi}$-*bounded* graph classes. A class of graph $\mathcal{G}$ is said to be $\overrightarrow{\chi}$-*bounded* if there is a function f such that $\overrightarrow{\chi}(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Because $\chi \leq \overrightarrow{\chi}$, a $\overrightarrow{\chi}$-bounded graph class is also χ-bounded. Conversely, one might wonder which χ-bounded graph classes are also $\overrightarrow{\chi}$-bounded.

The χ-boundedness of graph classes defined by forbidden induced subgraphs have been particularly investigated. For a fixed graph F, let us denote by $\mathrm{Forb}(F)$ the class of graphs that do not contain F as an induced subgraph. Erdős [6] showed that there are graphs with arbitrarily high girth and chromatic number. This implies that if F contains a cycle, then $\mathrm{Forb}(F)$ is not χ-bounded. Conversely, Gyárfás [7] and Sumner [8] independently made the following beautiful and difficult conjecture

Conjecture 24 ([7] and [8]). *For every tree T, the class $\mathrm{Forb}(T)$ is χ-bounded.*

It is natural to ask whether this conjecture generalizes to proper orientations.

Problem 25. *Is the class $\mathrm{Forb}(T)$ $\overrightarrow{\chi}$-bounded for all tree T ?*

Gyárfás [5] establishes Conjecture 24 for stars by showing that a graph in $\mathrm{Forb}(K_{1,n})$ has maximum degree $R(n, \omega(G))$, where $R(p, q)$ denotes the Ramsey number (p, q). In particular, this shows that $\mathrm{Forb}(K_{1,n})$ is also $\overrightarrow{\chi}$-bounded.

In particular, if G is a planar claw-free graph (recall that the claw is the graph $K_{1,3}$), Gyárfás result gives us that $\overrightarrow{\chi}(G) \leq \Delta(G) \leq R(3, 4) = 9$. This is also a partial answer to whether planar graphs have bounded proper orientation number. However, this bound is not tight, as we show next. In [9], Plummer showed that any claw-free 3-connected planar graph has maximum degree at most 6. His result can be extended to any claw-free planar graph.

Theorem 26. *If G is a claw-free planar graph, then $\Delta(G) \leq 6$.*

Proof. The proof is by induction on the number of vertices of G. If G is disconnected, then, by the induction hypothesis, each connected component of G has maximum degree at most 6 and so $\Delta(G) \leq 6$.

Assume that G has a cut-vertex u. As G is claw-free, $G - u$ has exactly two components C_i, $i = 1, 2$, and the neighborhood of u in each C_i is a clique N_i. Observe that $N_i \cup \{u\}$ is a clique, which has size at most 4 because G is planar, so $|N_i| \leq 3$. Hence $d(u) = |N_1| + |N_2| \leq 6$. Now by the induction hypothesis applied to $G[V(C_1) \cup \{u\}]$ and $G[V(C_2) \cup \{u\}]$, we obtain that every vertex distinct from u has degree at most 6. Therefore $\Delta(G) \leq 6$. Henceforth we may assume that G is 2-connected.

Assume that G has a 2-cut $\{u, v\}$ (that is $G - \{u, v\}$ is disconnected). The graph $G' = G - v$ is connected with cut-vertex u. As above, $G' - u$ has exactly two components, C_1 and C_2, and $N_i = N(u) \cap C_i$ is a clique, for $i = 1, 2$ of size at most 3. We claim that $d(u) \leq 6$. If $uv \notin E(G)$, then $d(u) = |N_1| + |N_2|$, so $d(u) \leq 6$. If $uv \in E(G)$, then $d(u) = |N_1| + |N_2| + 1$. But $|N_1| + |N_2| \leq 5$ for otherwise there exist $u_1 \in N_1$ and $u_2 \in N_2$ non-adjacent to v (because G has no clique of size 5), so $G[\{u, v, u_1, u_2\}]$ is a claw, a contradiction. Therefore $d(u) \leq 6$. Similarly, one proves $d(v) \leq 6$. Now by the induction hypothesis applied to $G[V(C) \cup \{u, v\}]$ for each connected component of $G - \{u, v\}$, we obtain that every vertex distinct from u and v has degree at most 6; hence $\Delta(G) \leq 6$.

Henceforth, we may assume that G is 3-connected and the result follows by Plummer [9]. $\square$

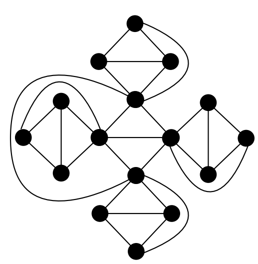


Figure 10: A planar claw-free graph G^* with maximum degree 6 and proper orientation number 6.

Theorem 26 is tight as shown by the graph G depicted in Figure 10 which is claw-free, planar and has maximum degree 6. Moreover, Theorem 26 implies that every planar claw-free graph has proper-orientation number at most 6. This is tight as shown by the following proposition.

Proposition 27. *The graph G^*, depicted in Figure 10, has proper orientation number equal to 6.*

Proof. The graph G^* is made of 5 blocks isomorphic to K_4. One one them (in the center of the figure), denoted by C intersects the four others. For every vertex

v of C, let $B(v)$ be the block intersecting C in v. Assume for a contradiction that G has a proper 5-orientation D. There are two vertices v_1 and v_2 in C, such that $d_D^-(v_i) \in \{0, 1, 2, 3\}$. Now the set of in-degrees of the other vertices of $B(v_i)$ is exactly $\{0, 1, 2, 3\} \setminus \{d_D^-(v_i)\}$. Thus inside $B(v)$ there are exactly $6 - (0 + 1 + 2 + 3 - d_D^-(v_i)) = d_D^-(v_i)$ arcs towards v. Hence all the edges such that v_i is an endpoint are oriented from v_i to its neighbors in C. This is a contradiction, because the edge $v_1 v_2$ cannot be oriented both ways. □

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